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# A power of an entire function sharing one value with its derivative

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ABSTRACT

In this paper, we investigate uniqueness problems of entire functions that share one value with one of their derivatives. Let $f$ be a non-constant entire function, $n$ and $k$ be positive integers. If $f^n$ and $(f^n)^{(k)}$ share 1 CM and $n \geq k+1$, then $f^n = (f^n)^{(k)}$, and $f$ assumes the form $f(z) = ce^{\frac{\lambda}{n}z}$, where $c$ is a non-zero constant and $\lambda^k = 1$. This result shows that a conjecture given by Brück is true when $F = f^n$, where $n \geq 2$ is an integer.


## 1. Introduction

In what follows, a meromorphic (resp. entire) function always means a function which is meromorphic (resp. analytic) in the whole complex plane. We will use the standard notation in Nevanlinna's value distribution theory of meromorphic functions; see, e.g. [1].

We say that two meromorphic functions $f$ and $g$ share $a \in \mathbb{C}$ IM (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that $f$ and $g$ share $a$ CM (counting multiplicities). Let $m$ and $p$ be positive integers. We denote by $N_p\left(r, 1/(f-a)\right)$ the counting function of the zeros of $f-a$ where $m$-fold zeros are counted $m$ times if $m \leq p$ and $p$ times if $m > p$.

Recently, a widely studied subtopic of the uniqueness theory has been the consideration of shared value problems relative to a meromorphic function $F$ and its $k$th derivative $F^{(k)}$. In order to get the uniqueness of sharing one value of $F$ and $F^{(k)}$, some deficient assumption is needed. The reader is invited to see the recent papers [2–7].

The purpose of this paper is to study a power of an entire function sharing one value with its derivative. We will give some results concerning Brück's Conjecture, which is mentioned later.

Let $f$ be a non-constant entire function and $n$ be a positive integer. If $f^n$ and $(f^n)'$ share 1 CM, then there exists an entire function $\alpha$ such that

$$\frac{(f^n)'-1}{f^n-1} = e^{\alpha}.$$

Rewriting above equation, we have

$$g_1 + g_2 + g_3 = 1, \tag{1.1}$$

where $g_1 = (f^n)'$, $g_2 = -e^{\alpha} f^n$, $g_3 = e^{\alpha}$.

There are many results on a combination of three meromorphic functions

$$f_1 + f_2 + f_3 = 1 \tag{1.2}$$

in uniqueness theory. The following result is a useful one. As for the proof; see, e.g. [8].

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**Theorem 1.1.** *Let $f_j$ $(j = 1, 2, 3)$ be meromorphic functions satisfying (1.2). If $f_1$ is not a constant, and*

$$\sum_{j=1}^{3} N_2(r, 1/f_j) + \sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r), \quad r \in I,$$

*where $\lambda < 1$, $T(r) = \max\{T(r, f_1), T(r, f_2), T(r, f_3)\}$, $I$ denotes a set of $r \in (0, \infty)$ with infinite linear measure. Then either $f_2 = 1$ or $f_3 = 1$.*

Applying Theorem 1.1 on (1.1), the present authors [8] got:

**Theorem 1.2.** *Let $f$ be a non-constant entire function, $n \geq 7$ be an integer. If $f^n$ and $(f^n)'$ share 1 CM and, then $f^n = (f^n)'$, and $f$ assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

*where $c$ is a non-zero constant.*

It is natural to ask whether $n$ can be reduced in Theorem 1.2. In fact, there are much more relations between $g_j$ in (1.1) than $f_j$ in (1.2). By studying this, we give a result improving Theorem 1.2 in Section 2. In Section 3, we consider a power of an entire function sharing 1 IM with its derivative. We provide some concluding remarks in Section 4.

## 2. Sharing 1 CM

In order to get a general result, we consider $f^n$ sharing 1 CM with its $k$th derivative, where $k$ is a positive integer, and obtain the following result:

**Theorem 2.1.** *Let $f$ be a non-constant entire function, $n$ and $k$ be positive integers. If $f^n$ and $(f^n)^{(k)}$ share 1 CM and $n \geq k + 1$, then $f^n = (f^n)^{(k)}$, and $f$ assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z}, \tag{2.1}$$

*where $c$ is a non-zero constant and $\lambda^k = 1$.*

In order to prove Theorem 2.1, we need the following lemma.

**Lemma 2.2** ([1, Lemma 4.3]). *Let $f$ be a non-constant meromorphic function and $P(f)$ be a polynomial in $f$ with constant coefficients. Let $b_j$ $(j = 1, \ldots, q)$ be distinct finite values. If $q > \deg P$, then*

$$m\left(r, \frac{P(f)f'}{(f - b_1)(f - b_2)\cdots(f - b_q)}\right) = S(r, f).$$

We begin to prove Theorem 2.1:

**Proof.** Denote

$$F = f^n. \tag{2.2}$$

Since $F$ and $F^{(k)}$ share 1 CM, then there exists an entire function $\alpha$, such that

$$F^{(k)} - 1 = e^{\alpha}(F - 1). \tag{2.3}$$

Suppose first that $e^{\alpha}$ is a non-constant entire function. By differentiation, we have

$$F^{(k+1)} = \alpha' e^{\alpha}(F - 1) + e^{\alpha}F'. \tag{2.4}$$

Combining (2.3) with (2.4) yields

$$F^{(k+1)}F - \alpha' F^{(k)}F - F^{(k)}F' = F^{(k+1)} - \alpha'(F^{(k)} + F) - F' + \alpha'. \tag{2.5}$$

By induction, we deduce from (2.2) that

$$F^{(k)} = \sum_{\lambda} c_\lambda f^{l_0^\lambda}(f')^{l_1^\lambda} \cdots \left(f^{(k)}\right)^{l_k^\lambda}, \tag{2.6}$$

where $l_0^\lambda, \ldots, l_k^\lambda$ are non-negative integers satisfying $\sum_{j=0}^{k} l_j^\lambda = n$, $n - k \leq l_0^\lambda \leq n - 1$ and $c_\lambda$ are constants.

Substituting (2.2) and (2.6) into (2.5), we have

$$f^n \cdot f^{n-k-1}P = Q, \tag{2.7}$$

where $Q$ is a differential polynomial in $f$ of the degree $n$, $P$ is a differential polynomial in $f$ of the degree $k+1$ and the coefficients of $P$ are the polynomials in $\alpha'$. In particular, every monomial of $P$ has the form

$$R(\alpha')f^{l_0}(f')^{l_1}\cdots\left(f^{(k+1)}\right)^{l_{k+1}},$$

where $l_0,\ldots,l_{k+1}$ are non-negative integers satisfying $\sum_{j=0}^{k+1} l_j = k+1$ and $l_0 \le k$ (since $n \ge k+1$), $R(\alpha')$ is a polynomial in $\alpha'$ with constant coefficients. From this and logarithmic derivative lemma, we obtain

$$m\left(r, \frac{P}{f^k f'}\right) = S(r,f). \tag{2.8}$$

If $P \not\equiv 0$, we get from (2.7) and Clunie lemma (for the proof, see, e.g. [9, Chapter 2.4]) that

$$T(r, f^{n-k-1}P) = m(r, f^{n-k-1}P) = S(r,f).$$

Combining this with (2.8), we have

$$\begin{aligned} m\left(r, \frac{1}{f^{n-1}f'}\right) &\le m\left(r, \frac{f^{n-k-1}P}{f^{n-1}f'}\right) + m\left(r, \frac{1}{f^{n-k-1}P}\right) \\ &\le m\left(r, \frac{P}{f^k f'}\right) + T(r, f^{n-k-1}P) + O(1) \\ &= S(r,f). \end{aligned}$$

From the above inequality and Lemma 2.2, we have

$$\begin{aligned} m\left(r, \frac{1}{f}\right) &= \frac{1}{n} m\left(r, \frac{1}{f^n}\right) \\ &\le \frac{1}{n} m\left(r, \frac{f^{n-1}f'}{f^n}\right) + \frac{1}{n} m\left(r, \frac{1}{f^{n-1}f'}\right) \\ &\le S(r,f) \end{aligned} \tag{2.9}$$

and

$$m\left(r, \frac{1}{f^n - 1}\right) \le m\left(r, \frac{f^{n-1}f'}{f^n - 1}\right) + m\left(r, \frac{1}{f^{n-1}f'}\right) = S(r,f). \tag{2.10}$$

From (2.3) and (2.10), we get

$$m(r, e^{\alpha}) \le m\left(r, \frac{F^{(k)}}{F-1}\right) + m\left(r, \frac{1}{f^n - 1}\right) + O(1) \le S(r,f),$$

which means that $T(r, e^{\alpha}) = S(r,f)$.

Rewriting (2.3), yields

$$e^{\alpha} - 1 = \frac{F^{(k)} - F}{F - 1} = \frac{f^{n-k}(P_k(f) + f^k)}{f^n - 1},$$

where $P_k(f)$ is a differential polynomial in $f$. Noting that $n \ge k+1$, we get

$$N\left(r, \frac{1}{f}\right) \le N\left(r, \frac{1}{e^{\alpha} - 1}\right) \le T(r, e^{\alpha}) + O(1) = S(r,f).$$

Combining this with (2.9), we obtain

$$T(r,f) = T\left(r, \frac{1}{f}\right) + O(1) = m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + O(1) = S(r,f),$$

which is a contradiction. Hence $P \equiv 0$. Then $Q \equiv 0$ from (2.7), where $Q = F^{(k+1)} - \alpha'(F^{(k)} + F) - F' + \alpha'$. We get from (2.5) that $F^{(k+1)}F - \alpha' F^{(k)}F - F^{(k)}F' = 0$. If $F$ is a polynomial, then $F-1$ and $F^{(k)}-1$ cannot have the same zeros with the same multiplicities. Thus $FF^{(k)} \not\equiv 0$. Therefore

$$\frac{F^{(k+1)}}{F^{(k)}} = \alpha' + \frac{F'}{F}.$$

By integration, we have $F^{(k)} = \mathrm{d}F\mathrm{e}^{\alpha}$, where $d$ is a non-zero constant. Substituting this and (2.2) into (2.3), we have

$$(d-1)f^n = \frac{1-\mathrm{e}^{\alpha}}{\mathrm{e}^{\alpha}}.$$

Obviously, $d \neq 1$ and all zeros of $1-\mathrm{e}^{\alpha}$ have the multiplicities at least $n$. Noting that $n \geq 2$, we get from the second fundamental theorem that

$$\begin{aligned} T(r,\mathrm{e}^{\alpha}) &\leq \overline{N}(r,\mathrm{e}^{\alpha}) + \overline{N}\left(r,\frac{1}{\mathrm{e}^{\alpha}}\right) + \overline{N}\left(r,\frac{1}{\mathrm{e}^{\alpha}-1}\right) + S(r,\mathrm{e}^{\alpha}) \\ &\leq \frac{1}{n}N\left(r,\frac{1}{\mathrm{e}^{\alpha}-1}\right) + S(r,\mathrm{e}^{\alpha}) \\ &\leq \frac{1}{n}T(r,\mathrm{e}^{\alpha}) + S(r,\mathrm{e}^{\alpha}), \end{aligned}$$

which is a contradiction since we suppose first that $\mathrm{e}^{\alpha}$ is a non-constant entire function.

Suppose then that $\mathrm{e}^{\alpha}$ is a non-zero constant. Say $A$. From (2.3), we have

$$F^{(k)} - AF = 1 - A. \tag{2.11}$$

If $A \neq 1$, we claim that 0 is a Picard exceptional value of $f$. Otherwise, suppose that $z_0$ is a zero of $f$ of the multiplicity $p$. Noting that $n \geq k+1$, $z_0$ is zero of $F^{(k)}$ of the multiplicity $np-k$. Then we get $A = 1$ from (2.11), which is a contradiction. We may assume that $f = \mathrm{e}^{\beta}$, where $\beta$ is a non-constant entire function. Substituting this into (2.11), we obtain

$$\left(P(\beta') - A\right)\mathrm{e}^{n\beta} = 1 - A,$$

where $P(\beta')$ is a differential polynomial in $\beta'$. Obviously, $P(\beta') \not\equiv A$. Then $nT(r,\mathrm{e}^{\beta}) = T(r,\mathrm{e}^{n\beta}) = T(r,(1-A)/(P(\beta')-A)) = T(r,P(\beta')) + O(1) = S(r,\mathrm{e}^{\beta})$ from the above equation, which contradicts with the fact that $\beta$ is a non-constant entire function.

Hence $A = 1$. Therefore, $F = F^{(k)}$ from (2.11). By the same arguments as above, we have that 0 is a Picard exceptional value of $f$, then $f$ assumes the form

$$f(z) = c\mathrm{e}^{\frac{\lambda}{n}z},$$

where $c$ is a non-zero constant and $\lambda^k = 1$. □

For the special case $k = 1$, we have the following corollary improving Theorem 1.2:

**Corollary 2.3.** *Let $f$ be a non-constant entire function, $n$ $(\geq 2)$ be an integer. If $f^n$ and $(f^n)'$ share 1 CM, then $f^n = (f^n)'$, and $f$ assumes the form*

$$f(z) = c\mathrm{e}^{\frac{1}{n}z}, \tag{2.12}$$

*where $c$ is a non-zero constant.*

**Example 2.4.** Let $f$ be a non-constant solution of

$$\frac{f'-1}{f-1} = \mathrm{e}^{z}.$$

Then $f$ and $f'$ share 1 CM, while $f \neq f'$. This example shows that the assumption $n \geq 2$ in Corollary 2.3 is sharp.

## 3. Sharing 1 IM

In this section, we consider a power of an entire function sharing 1 IM with its $k$th derivative:

**Theorem 3.1.** *Suppose that $f$ is an entire function, $n$ and $k$ are positive integers satisfying $n \geq k+2$. If $f^n$ and $(f^n)^{(k)}$ share 1 IM, then $f^n = (f^n)^{(k)}$, and $f$ assumes the form (2.1).*

**Proof.** Suppose that $F \not\equiv F^{(k)}$. From the second fundamental theorem, we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,1/F) + \overline{N}(r,1/(F-1)) + S(r,F) \\ &\leq \overline{N}(r,1/f) + \overline{N}\left(r,1/\left(F^{(k)}/F-1\right)\right) + S(r,f) \\ &\leq \overline{N}(r,1/f) + T\left(r,F^{(k)}/F\right) + S(r,f) \\ &= \overline{N}(r,1/f) + N\left(r,F^{(k)}/F\right) + S(r,f) \end{aligned}$$

$$
\begin{aligned}
&\leq \overline{N}(r, 1/f) + N_k(r, 1/F) + S(r,f) \\
&\leq (k+1)\overline{N}(r, 1/f) + S(r,f) \\
&\leq (k+1)T(r,f) + S(r,f),
\end{aligned}
$$

which contradicts with $n \geq k+2$. Thus $F = F^{(k)}$. Using the same way as in the proof of Theorem 2.1, we get that $f$ assumes the form (2.1). □

Comparing Theorem 2.1 with Theorem 3.1, we give an open problem as follows:

**Question 1.** *What happens if $n \geq k+2$ is replaced by $n \geq k+1$ in Theorem 3.1?*

In this paper, we give an answer to Question 1 when $k = 1$ by the following result, which also improves Corollary 2.3.

**Theorem 3.2.** *Let $f$ be a non-constant entire function, $n(\geq 2)$ be an integer. If $f^n$ and $(f^n)'$ share 1 IM, then $f^n = (f^n)'$, and $f$ assumes the form (2.12).*

**Proof.** Let $F$ be given by (2.2). Since $F$ and $F'$ share 1 IM, we know that all zeros of $F-1$ are simple zeros. Suppose that $F \not\equiv F'$. Denote

$$H := \frac{F'(F'-F)}{F(F-1)} = \frac{nf^{n-2}f'(nf'-f)}{f^n-1}. \tag{3.1}$$

Then $H$ is an entire function and

$$
\begin{aligned}
T(r,H) &= m(r,H) = m\left(r, \frac{F'}{F-1}\left(\frac{F'}{F}-1\right)\right) \\
&\leq m\left(r, \frac{F'}{F-1}\right) + m\left(r, \frac{F'}{F}\right) + O(1) \\
&= S(r,f).
\end{aligned} \tag{3.2}
$$

Rewriting (3.1) gives

$$F'^2 - F'F = H(F^2 - F).$$

Differentiating twice, we obtain

$$2F'F'' - F'^2 - FF'' = H'(F^2-F) + H(2FF' - F') \tag{3.3}$$

and

$$2F''^2 + 2F'F''' - 3F'F'' - FF''' = H''(F^2-F) + 2H'(2FF'-F') + H(2F'^2 + 2FF'' - F''). \tag{3.4}$$

Let $z_1$ be a zero of $F-1$. Then $F(z_1) = F'(z_1) = 1$. From (3.3) and (3.4) we have

$$
\begin{aligned}
&F''(z_1) = H(z_1) + 1, \\
&F'''(z_1) = 2H'(z_1) - H^2(z_1) + 2H(z_1) + 1.
\end{aligned}
$$

Set

$$\phi = \frac{F'' - (H+1)F'}{F-1}, \tag{3.5}$$

$$\psi = \frac{F''' - (2H' - H^2 + 2H + 1)F'}{F-1}. \tag{3.6}$$

Then $\phi$ and $\psi$ are entire functions since all zeros of $F-1$ are simple. Hence, we get from (3.2) that

$$
\begin{aligned}
T(r,\phi) &= m(r,\phi) \\
&\leq m\left(r, \frac{F''}{F-1}\right) + m\left(r, \frac{F'}{F-1}\right) + m(r,H) + O(1) \\
&= S(r,f).
\end{aligned}
$$

Similarly,

$$T(r,\psi) = S(r,f).$$

From (3.5), we obtain

$$F'' = (H+1)F' + \phi(F-1). \tag{3.7}$$

Differentiating the above equation gives

$$F''' = H'F' + (H+1)F'' + \phi'(F-1) + \phi F'. \tag{3.8}$$

Combining (3.7), (3.8) and (3.6) yields

$$F'(2H^2 - H' + \phi) = (F-1)\left(\psi - \phi' - (1-H)\phi\right).$$

Namely,

$$nf^{n-1}f'(2H^2 - H' + \phi) = (f^n - 1)\left(\psi - \phi' - (1-H)\phi\right).$$

If $2H^2 - H' + \phi \neq 0$, the last two equations imply

$$N\left(r, \frac{1}{F-1}\right) \leq N\left(r, \frac{1}{2H^2 - H' + \phi}\right) = S(r,f),$$
$$N\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{\psi - \phi' - (1-H)\phi}\right) = S(r,f).$$

By the second fundamental theorem, we obtain

$$T(r, f^n) \leq \overline{N}\left(r, \frac{1}{f^n - 1}\right) + \overline{N}\left(r, \frac{1}{f}\right) + S(r,f) = S(r,f),$$

which is a contradiction. Therefore

$$2H^2 - H' + \phi = 0. \tag{3.9}$$

Let $z_0$ be a zero of $f$. Then $F(z_0) = F'(z_0) = 0$ since $n \geq 2$. Substituting this into (3.4) and (3.5), we get

$$F''(z_0)(2F''(z_0) + H(z_0)) = 0, \tag{3.10}$$

$$F''(z_0) = -\phi(z_0). \tag{3.11}$$

We claim that

$$2F''(z_0) = -H(z_0). \tag{3.12}$$

In fact, if $n \geq 3$, then $F''(z_0) = 0$. Furthermore, we get from (3.1) that $H(z_0) = 0$. Hence (3.12) holds. If $n = 2$, then $F''(z_0) = 2f'^2(z_0) + 2f(z_0)f''(z_0) = 2f'^2(z_0)$. If $F''(z_0) = 0$, then $f'(z_0) = 0$. From (3.1), we get $H(z_0) = 0$, and so (3.12) holds. If $F''(z_0) \neq 0$, (3.12) comes immediately from (3.10).

Substituting (3.11) and (3.12) into (3.9), we obtain

$$2H^2(z_0) + \frac{1}{2}H(z_0) - H'(z_0) = 0. \tag{3.13}$$

If $2H^2 + \frac{1}{2}H - H' \neq 0$, we get from (3.2) and (3.13) that

$$\overline{N}\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{2H^2 + \frac{1}{2}H - H'}\right) = S(r,f).$$

Noting that

$$\begin{aligned} N\left(r, \frac{1}{F-1}\right) &\leq N\left(r, \frac{1}{\frac{F'}{F} - 1}\right) \leq T\left(r, \frac{F'}{F}\right) + O(1) \\ &= N\left(r, \frac{F'}{F}\right) + m\left(r, \frac{F'}{F}\right) + O(1) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + S(r,f) \\ &= S(r,f). \end{aligned}$$

By the second fundamental theorem, we have a contradiction. Hence

$$2H^2 + \frac{1}{2}H - H' = 0. \tag{3.14}$$

Since $F \not\equiv F'$, we have $H \not\equiv 0$. Then

$$2H = \frac{H'}{H} - \frac{1}{2}.$$

Noting that $H$ is an entire function, we have

$$T(r,H) = m(r,H) \leq m\left(r, \frac{H'}{H}\right) + O(1) = S(r,H),$$

which means that $H$ is a constant. From (3.14), we know that $H = -\frac{1}{4}$. From (3.1), we obtain

$$(2F' - F)^2 = F.$$

Set

$$\gamma = 2F' - F \quad \text{or} \quad -\gamma = 2F' - F.$$

Then $F = \gamma^2$ and $F' = 2\gamma\gamma'$. Thus $4\gamma' = \gamma + 1$ or $4\gamma' = \gamma - 1$. If $4\gamma' = \gamma + 1$, by integration,

$$\gamma = Ae^{\frac{1}{4}z} - 1,$$

where $A$ is a non-zero constant. Let $z^* = 4\pi\mathrm{i} - 4\log A$. Then $\gamma(z^*) = -2$ and $\gamma'(z^*) = -\frac{1}{4}$. Thus $F'(z^*) = 1$ and $F(z^*) = 4$, which contradicts with $F$ and $F'$ sharing 1 IM. If $4\gamma' = \gamma - 1$, by integration, $\gamma = Be^{\frac{1}{4}z} + 1$, where $B$ is a non-zero constant. Let $z^* = -4\log A$. We obtain a contradiction by the same reasoning. Therefore, $F = F'$, and there exists a non-zero constant $c$ such that $f = ce^{\frac{1}{n}z}$. This completes the proof of Theorem 3.2. □

## 4. Concluding remarks

Now, we introduce the definition of weighted sharing: let $l$ be a non-negative integer or infinite. For any $a \in \mathbb{C}\bigcup\{\infty\}$, we denote by $E_l(a,f)$ the set of all $a$-points of $f$ where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq l$ and $l+1$ times if $m > l$. If $E_l(a,f) = E_l(a,g)$, we say that $f$ and $g$ share the value $a$ with weight $l$ (see [5]).

We say that $f$ and $g$ share $(a,l)$ if $f$ and $g$ share the value $a$ with weight $l$. It is easy to see that $f$ and $g$ share $(a,l)$ implies $f$ and $g$ share $(a,p)$ for $0 \leq p \leq l$. Also we note that $f$ and $g$ share a value $a$ IM or CM if and only if $f$ and $g$ share $(a,0)$ or $(a,\infty)$ respectively.

We recall the following result which is a corollary of Theorem 1.1 in [10]:

**Theorem 4.1.** *Let $F$ be a non-constant entire function and $k$ be a positive integer. Suppose that $F$ and $F^{(k)}$ share $(1,2)$. If*

$$\delta_2(0,F) + \delta_{2+k}(0,F) > 1,$$

*where $\delta_p(0,F) = 1 - \limsup_{r\to\infty} \frac{N_p(r,1/F)}{T(r,F)}$, then $F = F^{(k)}$.*

If $F = f^n$, where $f$ is a non-constant entire function and $n$ is a positive integer. Then

$$\delta_p(0,F) = 1 - \limsup_{r\to\infty} \frac{N_p(r,1/F)}{T(r,F)} \geq 1 - \frac{p}{n}.$$

Noting this, from Theorem 4.1 we have the following corollary:

**Corollary 4.2.** *Let $f$ be a non-constant entire function and $n, k$ be two positive integers. Denote $F = f^n$. Suppose that $F$ and $F^{(k)}$ share 1 CM. If $n \geq k+5$, then $F = F^{(k)}$.*

Obviously, Theorem 2.1 improves Corollary 4.2.

Without deficient assumption, Brück [11] proposed the following conjecture:

**Brück Conjecture 1.** *Let $F$ be a non-constant entire function. Suppose that*

$$\rho_2(F) := \limsup_{r\to\infty} \frac{\log^+\log^+ T(r,F)}{\log r}$$

*is not a positive integer or infinite. If $F$ and $F'$ share a finite value $b$ CM, then*

$$\frac{F' - b}{F - b} = c$$

*for some non-zero constant $c$.*

The conjecture has been verified in special cases only: (1) $\rho_2(F) < \frac{1}{2}$, see [12]; (2) $b = 0$, see [11]; (3) $N(r, 1/F') = S(r,F)$, see [11].

In this paper, Corollary 2.3 tells us Conjecture 1 is true when $F = f^n$, where $n \geq 2$ is an integer.

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